

BERNSTEIN POLYNOMIALS AND n -COPULAS

M. D. TAYLOR

Abstract. We give derivations of some basic results for the Bernstein approximation in n variables that are useful in investigating copulas. It is shown that Bernstein approximations of copulas are again copulas. We exhibit a stochastic interpretation for the Bernstein approximation of a copula.

1. Introduction

Bernstein approximations of 2-copulas were introduced and studied in [3] and [4]. We assume the reader is familiar with copulas; see, for example, [7] or [6].

This note was written to clarify for myself and my colleagues certain properties of Bernstein approximations that are useful in investigating copulas. We derive some of the basic properties of the Bernstein approximation for functions of n variables and then show that the Bernstein approximation of a copula is again a copula. Our most significant result is a stochastic interpretation of the Bernstein approximation of a copula. This interpretation was communicated to us by J. H. B. Kemperman in [2] for 2-copulas and we are not aware of its publication elsewhere.

The encouragement and contributions of our colleagues P. Mikusiński and X. Li to this note were crucial.

2. Bernstein polynomials and approximations

It is our convention that $I = [0, 1]$.

Definition 1. The m -th degree Bernstein polynomial $b_{i,m} : I \rightarrow \mathbb{R}$ is given by

$$b_{i,m}(t) = \binom{m}{i} t^i (1-t)^{m-i},$$

$i = 0, 1, \dots, m$. We extend this to $B_{i,m}^n : I^n \rightarrow \mathbb{R}$ by taking i to be a multi-index, $i = (i_1, \dots, i_n)$, where each $i_k \in \{0, 1, \dots, m\}$ and setting

$$B_{i,m}^n(x) = b_{i_1,m}(x_1) b_{i_2,m}(x_2) \cdots b_{i_n,m}(x_n)$$

where $x = (x_1, \dots, x_n) \in I^n$.

Notice that $\{B_{i,m}^n\}$ is a partition of unity over I^n .

Here is the intuition behind the Bernstein polynomial: Consider the act of tossing a coin m times with probability of heads on each toss being x . This scenario can be represented by a random vector $X : \Omega \rightarrow \{0, 1\}^m$ with the property that if $X(\omega) =$

Date: March 5, 2009.

2000 Mathematics Subject Classification. Primary: 60E05; Secondary: 62E17 62H99.

Key words and phrases. Copulas, Bernstein polynomials.

(x_1, \dots, x_m) , then $P(x_i = 1) = x$. We then introduce the random variable Y defined by

$$Y(\omega) = \sum \{x_i : x_i = 1\},$$

in other words, the number of heads that were tossed. This is familiarly described as a binomially distributed random variable with parameters m and x . It is easily shown that

$$E(Y) = mx \quad \text{and} \quad \text{Var}(Y) = E((Y - E(Y))^2) = mx(1 - x).$$

Notice that

$$b_{i,m}(x) = P(Y = i).$$

Definition 2. If $f : I^n \rightarrow \mathbb{R}$, we define the Bernstein approximation to f to be

$$\mathcal{B}_m^n(f) = \sum_i f\left(\frac{i}{m}\right) B_{i,m}^n$$

where i ranges over all multi-indices $i = (i_1, \dots, i_n)$ such that each $i_k \in \{0, 1, \dots, m\}$, and by $\frac{i}{m}$ we mean the vector $(\frac{i_1}{m}, \dots, \frac{i_n}{m})$.

It can be shown by induction that

$$\int_{I^n} B_{i,m}^n d\lambda^n = \frac{1}{(m+1)^n}$$

where λ^n is Lebesgue measure on \mathbb{R}^n .

3. The Weierstrass approximation theorem via Bernstein polynomials

Theorem 1. If $f : I^n \rightarrow \mathbb{R}$ is continuous, then $\mathcal{B}_m^n(f) \rightarrow f$ uniformly on I^n as $m \rightarrow \infty$.

Proof. Choose $\epsilon > 0$. Since f is uniformly continuous on I^n , there exists $\delta > 0$ with the property that if $x = (x_1, \dots, x_n)$, $y = (y_1, \dots, y_n)$, and $|x_i - y_i| < \delta$ for all i , then $|f(x) - f(y)| < \epsilon$. In what follows, it is convenient to define d by $d(x, y) = \max\{|x_1 - y_1|, \dots, |x_n - y_n|\}$.

Set $f_m = \mathcal{B}_m^n(f)$. We suppose that m is so large that $\frac{1}{4m\delta^2} < \epsilon$; we shall show that this makes $|f_m - f|$ "small." Choose $x = (x_1, \dots, x_n) \in I^n$. Then

$$|f_m(x) - f(x)| \leq \sum_{d(\frac{i}{m}, x) < \delta} |f(\frac{i}{m}) - f(x)| B_{i,m}^n(x) + \sum_{d(\frac{i}{m}, x) \geq \delta} |f(\frac{i}{m}) - f(x)| B_{i,m}^n(x)$$

where $i = (i_1, \dots, i_n)$, a multi-index. We see that

$$\sum_{d(\frac{i}{m}, x) < \delta} |f(\frac{i}{m}) - f(x)| B_{i,m}^n(x) < \epsilon$$

by uniform continuity of f . To find a bound for the other term, we first introduce independent, binomially distributed random variables X_1, \dots, X_n with parameters m and x . By Tchebycheff's inequality, for each $j = 0, 1, \dots, m$ we have

$$P\left(\left|\frac{X_j}{m} - x_j\right| \geq \delta\right) \leq \frac{x_j(1-x_j)}{m\delta^2} \leq \frac{1}{4m\delta^2} < \epsilon.$$

Let $M = \max f$. Then

$$\sum_{d(\frac{i}{m}, x) \geq \delta} |f(\frac{i}{m}) - f(x)| B_{i,m}^n(x) \leq 2M \sum_{d(\frac{i}{m}, x) \geq \delta} B_{i,m}^n(x).$$

We see that

$$\begin{aligned} \sum_{d(\frac{i}{m}, x) \geq \delta} B_{i,m}^n(x) &= \sum_{d(\frac{i}{m}, x) \geq \delta} P\left(\left(\frac{X_1}{m}, \dots, \frac{X_n}{m}\right) = \frac{i}{m}\right) \\ &\leq \sum_{j=1}^n P\left(\left|\frac{X_j}{m} - x_j\right| \geq \delta\right) < n\epsilon. \end{aligned}$$

Thus $|f_m(x) - f(x)| < \epsilon + 2Mn\epsilon$, and we are done. \square

4. Derivatives of Bernstein approximations

4.1. Derivatives of Bernstein polynomials. If $f : I^n \rightarrow \mathbb{R}$, set

$$(1) \quad f_m = \mathcal{B}_m^n(f) = \sum_i f\left(\frac{i}{m}\right) B_{i,m}^n = \sum_i f\left(\frac{i}{m}\right) b_{i_1,m} \otimes \dots \otimes b_{i_n,m}$$

where $i = (i_1, \dots, i_n)$ and $i_k = 0, 1, \dots, m$ and the symbolism $g \otimes h$ is interpreted to mean $(g \otimes h)(u, v) = g(u)h(v)$. We want to compute partial derivatives of f_m . In particular we want compute the mixed partial $\frac{\partial^n f_m}{\partial x_1 \dots \partial x_n}$ which we denote ∂f_m . In the case where f_m is a cumulative probability distribution function, ∂f_m is the associated probability density.

It is convenient at this point to introduce another notation. Let $g : A \rightarrow \mathbb{R}$ where $A \subseteq \mathbb{R}^n$. Suppose $v \in \mathbb{R}^n$ such that $A \cap (A - v) \neq \emptyset$. We then define a function $\Delta_v g : A \cap (A - v) \rightarrow \mathbb{R}$ by $\Delta_v g(p) = g(p + v) - g(p)$. That is, $\Delta_v g(p)$ is the variation of g starting at p in the direction v . Next, let e_1, \dots, e_n be the standard orthonormal basis for \mathbb{R}^n , that is, $e_1 = (1, 0, \dots, 0)$, $e_2 = (0, 1, 0, \dots, 0)$, etc. For $f : I^n \rightarrow \mathbb{R}$ and $i = (i_1, \dots, i_n)$, where $i_k = 0, 1, \dots, m-1$, we define

$$\Delta_{i,m}^n f = \Delta_{\frac{1}{m}e_1} \Delta_{\frac{1}{m}e_2} \dots \Delta_{\frac{1}{m}e_n} f\left(\frac{i}{m}\right).$$

We can think of $\Delta_{i,m}^n f$ as the variation of f over the n -dimensional square $[\frac{i}{m}, \frac{i+1}{m}) \times \dots \times [\frac{i_n}{m}, \frac{i_n+1}{m})$.

Returning to the problem of derivatives, one calculates

$$b'_{i,m} = \begin{cases} -m b_{0,m-1}, & i = 0, \\ m(b_{i-1,m-1} - b_{i,m-1}), & 0 < i < m, \\ m b_{m-1,m-1}, & i = m. \end{cases}$$

If we set $b_{-1,m-1} = b_{m,m-1} = 0$, then we may reduce this to

$$(2) \quad b'_{i,m} = m(b_{i-1,m-1} - b_{i,m-1}), \quad 0 \leq i \leq m.$$

If one then considers the case where $n = 1$ so that

$$f_m = \sum_{i=0}^m f\left(\frac{i}{m}\right) b_{i,m},$$

where i is now an integer, then one easily calculates

$$(3) \quad f'_m = m \sum_{j=0}^{m-1} \left(\Delta_{\frac{1}{m}e_1} f\left(\frac{j}{m}\right) \right) b_{j,m-1}.$$

We then pass to the general n -dimensional case where f_m has the form given in (1) and by repeatedly invoking the 1-dimensional case and Equation (3), we obtain

$$\partial f_m = m^n \sum_j (\Delta_{j,m}^n f) b_{j_1, m-1} \otimes \cdots \otimes b_{j_n, m-1}$$

where $j = (j_1, \dots, j_n)$ and $j_k = 0, 1, \dots, m-1$.

It is well-known that the Bernstein approximation of a copula is again a copula (see, for example, [3] and [4]), but this is also an immediate consequence of this last formula:

Theorem 2. *The Bernstein approximation of an n -copula is again an n -copula.*

Proof. Let $C_m = \mathcal{B}_m^n(C)$ where C is an n -copula. The boundary conditions for a copula are easily checked. The only questionable condition is whether or not C_m is n -increasing. But this follows from the fact that the terms of

$$\partial C_m = m^n \sum_j (\Delta_{j,m}^n C) b_{j_1, m-1} \otimes \cdots \otimes b_{j_n, m-1}$$

are nonnegative. \square

4.2. Some identities and estimates. In what follows, we assume that $x \in I$, $m = 1, 2, 3, \dots$, and $i = 0, 1, \dots, m$.

The following is straightforward to establish by induction over i :

Proposition 1.

$$\left(\frac{i}{m} - x\right) b_{i,m}(x) = x(1-x)(b_{i-1, m-1}(x) - b_{i, m-1}(x)).$$

Proposition 2.

$$\sum_{i=0}^m \left|x - \frac{i}{m}\right| |b_{i-1, m-1}(x) - b_{i, m-1}(x)| = \frac{1}{m}.$$

Proof. We assume that X is a binomially distributed random variable with parameters x and m and make use of Proposition 1:

$$\begin{aligned} \sum_{i=0}^m \left|x - \frac{i}{m}\right| |b_{i-1, m-1}(x) - b_{i, m-1}(x)| &= \frac{1}{x(1-x)} \sum_{i=0}^m \left(x - \frac{i}{m}\right)^2 b_{i,m}(x) \\ &= \frac{1}{x(1-x)} \frac{1}{m^2} \text{Var}(X) = \frac{1}{m}. \end{aligned}$$

\square

Proposition 3.

$$\sum_{i=0}^m \left|\frac{i}{m} - x\right| b_{i,m}(x) = 2x(1-x) b_{i_0, m-1}(x)$$

where $i_0 = \lfloor mx \rfloor$, the greatest integer less than or equal to mx .

Proof. Let us assume x is irrational, $0 < x < 1$. There is a unique nonnegative integer, namely $i_0 = \lfloor mx \rfloor$, such that

$$\frac{i_0}{m} < x < \frac{i_0 + 1}{m}.$$

We then perform a calculation in which we invoke Proposition 1:

$$\begin{aligned} \sum_{i=0}^m \left| \frac{i}{m} - x \right| b_{i,m}(x) &= \sum_{i=0}^{i_0} \left(x - \frac{i}{m} \right) b_{i,m}(x) + \sum_{i=i_0+1}^m \left(\frac{i}{m} - x \right) b_{i,m}(x) \\ &= x(1-x) \left(\sum_{i=0}^{i_0} (b_{i,m-1}(x) - b_{i-1,m-1}(x)) + \sum_{i=i_0+1}^m (b_{i-1,m-1}(x) - b_{i,m-1}(x)) \right) \\ &= 2x(1-x) b_{i_0,m-1}(x). \end{aligned}$$

We then obtain the proof for general x by invoking continuity. \square

A proof of the following has been shown to us informally by our colleague Xin Li, but it can also be found on p. 15 of [5].

Proposition 4.

$$\sum_{i=0}^m \left| \frac{i}{m} - x \right| b_{i,m}(x) = O\left(\frac{1}{\sqrt{m}}\right).$$

Proposition 5. For every $\delta > 0$ and $x \in I$,

$$\sum_{|x-i/m| \geq \delta} b_{i,m}(x) = o\left(\frac{1}{m}\right).$$

Proof. From [1, p. 304], we find that for each $\delta > 0$ and $s = 1, 2, \dots$, there exists $C = C(\delta, s)$ such that

$$\sum_{|x-i/m| \geq \delta} b_{i,m}(x) \leq C m^{-s}$$

for $m = 1, 2, \dots$ and $x \in I$. \square

4.3. Convergence of first derivatives of Bernstein approximations.

Theorem 3. Suppose that $f : I^n \rightarrow \mathbb{R}$ is a bounded function. Then for all $x = (x_1, \dots, x_n) \in (0, 1)^n$ at which f is differentiable and for $k = 1, \dots, n$, we have

$$\lim_{m \rightarrow \infty} \frac{\partial \mathcal{B}_m^n(f)}{\partial x_k}(x) = \frac{\partial f}{\partial x_k}(x).$$

Proof. Let us set

$$f_m = \mathcal{B}_m^n(f) = \sum_i f\left(\frac{i}{m}\right) b_{i_1,m} \otimes \cdots \otimes b_{i_n,m}$$

where $i = (i_1, \dots, i_n)$, each $i_k \in \{0, 1, \dots, m\}$, and

$$\sum_i = \sum_{i_1=0}^m \sum_{i_2=0}^m \cdots \sum_{i_n=0}^m.$$

We prove the proposition for the case $k = 1$.

First, we have

$$(4) \quad \frac{\partial f_m}{\partial x_1} = m \sum_i f\left(\frac{i}{m}\right) (b_{i_1-1,m-1} - b_{i_1,m-1}) \otimes b_{i_2,m} \otimes \cdots \otimes b_{i_n,m}.$$

Second, by the differentiability of f at x , we have

$$(5) \quad f\left(\frac{i}{m}\right) = f(x) + \sum_{k=1}^n \frac{\partial f}{\partial x_k}(x) \left(\frac{i_k}{m} - x_k\right) + \eta\left(\frac{i}{m} - x\right) \left|\frac{i}{m} - x\right|$$

where

$$\left|\frac{i}{m} - x\right| = \sqrt{\sum_{k=1}^n \left(\frac{i_k}{m} - x_k\right)^2}$$

and $\eta(s) \rightarrow 0$ as $s \rightarrow 0$ in \mathbb{R}^n . Next, making use of (5), it can be shown there is a constant M , dependent on x and n but independent of m , such that $|\eta(s)| \leq M$. This can be done by considering the case where s is “close” to 0 and the case where s is some fixed distance from 0.

Next, substituting from (5) into (4), we see that $(\partial f_m / \partial x_1)(x)$ becomes

$$m \sum_i \left(f(x) + \sum_{k=1}^n \frac{\partial f}{\partial x_k}(x) \left(\frac{i_k}{m} - x_k\right) + \eta\left(\frac{i}{m} - x\right) \left|\frac{i}{m} - x\right| \right) (b_{i_1-1, m-1}(x_1) - b_{i_1, m-1}(x_1)) b_{i_2, m}(x_2) \cdots b_{i_n, m}(x_n)$$

Now $\sum_{i_1} (b_{i_1-1, m-1}(x_1) - b_{i_1, m-1}(x_1)) = b_{-1, m-1}(x_1) - b_{m, m-1}(x_1) = 0$. Thus

$$\sum_i f(x) (b_{i_1-1, m-1}(x_1) - b_{i_1, m-1}(x_1)) b_{i_2, m}(x_2) \cdots b_{i_n, m}(x_n) = 0,$$

and for every $k > 1$ we have

$$\sum_i \frac{\partial f}{\partial x_k}(x) \left(\frac{i_k}{m} - x_k\right) (b_{i_1-1, m-1}(x_1) - b_{i_1, m-1}(x_1)) b_{i_2, m}(x_2) \cdots b_{i_n, m}(x_n) = 0.$$

On the other hand, it is easily seen that

$$\begin{aligned} m \sum_{i_1=0}^m \left(\frac{i_1}{m} - x_1\right) (b_{i_1-1, m-1}(x_1) - b_{i_1, m-1}(x_1)) \\ = \sum_{i_1} i_1 (b_{i_1-1, m-1}(x_1) - b_{i_1, m-1}(x_1)) = 1, \end{aligned}$$

and we know that $\sum_{i_k} b_{i_k, m}(x_k) = 1$ for $k > 1$, therefore

$$\begin{aligned} m \sum_i \frac{\partial f}{\partial x_1}(x) \left(\frac{i_1}{m} - x_1\right) (b_{i_1-1, m-1}(x_1) - b_{i_1, m-1}(x_1)) b_{i_2, m}(x_2) \cdots b_{i_n, m}(x_n) \\ = \frac{\partial f}{\partial x_1}(x). \end{aligned}$$

Thus we can write

$$\frac{\partial f_m}{\partial x_1}(x) = \frac{\partial f}{\partial x_1}(x) + S_m$$

where

$$S_m = m \sum_i \eta\left(\frac{i}{m} - x\right) \left|\frac{i}{m} - x\right| (b_{i_1-1, m-1}(x_1) - b_{i_1, m-1}(x_1)) b_{i_2, m}(x_2) \cdots b_{i_n, m}(x_n).$$

Our task now reduces to showing $S_m \rightarrow 0$. Choose $\epsilon > 0$. There exists $\delta > 0$ such that if $|s| < \delta$, then $|\eta(s)| < \epsilon$. Let us set $\delta_i = |(i/m) - x|$ and then break S_m into two pieces, $S_m = S_m^> + S_m^{\leq}$, where

$$S_m^< = \sum_{\delta_i < \delta} \quad \text{and} \quad S_m^{\geq} = \sum_{\delta_i \geq \delta}.$$

We first consider $S_m^<$. By Proposition 1,

$$b_{i_1-1, m-1}(x_1) - b_{i_1, m-1}(x_1) = \frac{\frac{i_1}{m} - x_1}{x_1(1-x_1)} b_{i_1, m}(x_1).$$

Then

$$\begin{aligned} |S_m^<| &\leq m\epsilon \sum_{\delta_i < \delta} \delta_i (b_{i_1-1, m-1}(x_1) - b_{i_1, m-1}(x_1)) b_{i_2, m}(x_2) \cdots b_{i_n, m}(x_n) \\ &\leq \frac{m\epsilon}{x_1(1-x_1)} \sum_{\delta_i < \delta} \delta_i \left| \frac{i_1}{m} - x_1 \right| b_{i_1, m}(x_1) b_{i_2, m}(x_2) \cdots b_{i_n, m}(x_n) \\ &\leq \frac{m\epsilon}{x_1(1-x_1)} \sum_i \sum_{k=1}^n \left| \frac{i_k}{m} - x_k \right| \left| \frac{i_1}{m} - x_1 \right| b_{i_1, m}(x_1) \cdots b_{i_n, m}(x_n). \end{aligned}$$

Now $\sum_{i_1} (i_1 - mx_1)^2 b_{i_1, m}(x_1)$ is the variance of a binomially distributed random variable, so

$$\sum_{i_1} \left(\frac{i_1}{m} - x_1 \right)^2 b_{i_1, m}(x_1) = \frac{x_1(1-x_1)}{m}.$$

On the other hand, for $k \neq 1$, we have by Proposition 4,

$$\sum_{i_1} \sum_{i_k} \left| \frac{i_1}{m} - x_1 \right| \left| \frac{i_k}{m} - x_k \right| b_{i_1, m}(x_1) b_{i_k, m}(x_k) = O\left(\frac{1}{\sqrt{m}}\right) O\left(\frac{1}{\sqrt{m}}\right) = O\left(\frac{1}{m}\right).$$

It follows that

$$|S_m^<| \leq \frac{m\epsilon}{x_1(1-x_1)} \frac{x_1(1-x_1)}{m} + m(n-1)\epsilon O\left(\frac{1}{m}\right) = \epsilon O(1).$$

Now we turn to S_m^{\geq} . We know the following:

- (1) $|\eta| \leq M$.
- (2) $\delta_i \leq \sqrt{n}$.
- (3) $\{i : \left| \frac{i}{m} - x \right| \geq \delta\} \subseteq \bigcup_{k=1}^n \{i : \left| \frac{i_k}{m} - x_k \right| \geq \frac{\delta}{n}\}.$

Using these facts plus Proposition 5, we obtain

$$\begin{aligned} |S_m^{\geq}| &\leq m M \sqrt{n} \sum_{\left| \frac{i}{m} - x \right| \geq \delta} b_{i_1, m}(x_1) \cdots b_{i_n, m}(x_n) \\ &\leq m M \sqrt{n} \sum_{\left| \frac{i_k}{m} - x_k \right| \geq \frac{\delta}{n}} \sum_{k=1}^n b_{i_k, m}(x_k) \\ &= m M \sqrt{n} n o\left(\frac{1}{m}\right) = o(1). \end{aligned}$$

We therefore conclude that

$$|S_m| \leq \epsilon O(1) + o(1) \rightarrow 0$$

as $m \rightarrow \infty$. □

5. A probabilistic interpretation of the Bernstein approximation of a copula

It is our goal here to construct random variables such that the Bernstein approximation is the cumulative distribution function of these new random variables. This probabilistic interpretation was brought to our attention by J. H. B. Kemperman in [2].

Let C be an n -copula. Suppose it is the cumulative distribution function of the ordered n -tuple of random variables (X_1, \dots, X_n) where each X_i is uniformly distributed over I . Let m be a “large” natural number and C_m be the $m \times \dots \times m$ Bernstein approximation of C ; that is

$$C_m(x_1, \dots, x_n) = \sum_{i_1, \dots, i_n=0}^m C\left(\frac{i_1}{m}, \dots, \frac{i_n}{m}\right) b_{i_1, m}(x_1) \cdots b_{i_n, m}(x_n)$$

where $x_1, \dots, x_n \in I$.

Next, for $i = 1, \dots, n$ and $j = 1, \dots, m$, we let X_i^j be independent random variables that are uniformly distributed over I and have the property that (X_1, \dots, X_n) and X_i^j are independent for all i, j . If it is helpful, we may regard these random variables as being defined over the space $I^n \times I^{mn}$ and having probability measure P where P has the form $\mu_C \times \lambda^{mn}$ and where it is understood that μ_C is the probability measure induced on I^n by C and λ^{mn} is Lebesgue measure on I^{mn} .

Now for each i , let $X_i^{(1)}, \dots, X_i^{(m)}$ be the order statistics for X_i^1, \dots, X_i^m . That is, whenever

$$X_i^{j_1} < \dots < X_i^{j_m},$$

then $X_i^{(k)} = X_i^{j_k}$.

It may be helpful to notice that by the independence and uniform distribution of our original random variables, for all i, j, r, s and all $x \in I$ we have

$$\begin{aligned} P(X_i^j = X_r^s) &= 0 \quad \text{if } X_i^j \neq X_r^s, \\ P(X_i^{(j)} = x) &= 0, \\ P(X_i^{(j)} = X_r^{(s)}) &= 0 \quad \text{if } X_i^{(j)} \neq X_r^{(s)}. \end{aligned}$$

Next, if k is the multi-index (k_1, \dots, k_n) where $k_r = 0, 1, \dots, m-1$, then we define

$$I_{m,k}^n = \left(\frac{k_1}{m}, \frac{k_1+1}{m}\right) \times \dots \times \left(\frac{k_n}{m}, \frac{k_n+1}{m}\right) = \left\{ \frac{k_r}{m} < X_r < \frac{k_r+1}{m} : r = 1, \dots, n \right\}.$$

This is a slight abuse of notation since X_r “lives” in $I^n \times I^{mn}$ rather than I^n , however the reader should easily make whatever mental adjustments are necessary in the arguments that follow. We then take χ_k to be the characteristic function of $I_{m,k}^n$ with the understanding that the domain of χ_k is I^n .

Finally we define

$$Y_r = \sum_{k_1, \dots, k_n=0}^{m-1} \chi_k(X_1, \dots, X_n) X_r^{(k_r)}$$

where $r = 1, \dots, n$ and $k = (k_1, \dots, k_n)$. We then have the following:

Theorem 4. $C_m(x_1, \dots, x_n) = P(Y_1 < x_1, \dots, Y_n < x_n)$ where $(x_1, \dots, x_n) \in I^n$.

Proof. For $x \in I$ and $i = 1, \dots, n$, we form a random variable $X_i^*(x)$ by setting

$$X_i^*(x) = \text{the number of } X_i^1, \dots, X_i^m \text{ less than } x.$$

We see that the following are true:

- (1) $X_i^*(x)$ takes values in $\{0, 1, \dots, m\}$.
- (2) $X_i^*(x)$ and $X_j^*(y)$ are independent for $i \neq j$.
- (3) $P(X_i^*(x) = k) = \binom{m}{k} x^k (1-x)^{m-k} = b_{i,m}(x)$.

We also see that

$$(6) \quad \begin{aligned} \{X_i^*(x) = 0\} &\stackrel{P\text{-a.e.}}{=} \{x < X_i^{(1)}\}, \\ \{X_i^*(x) = k\} &\stackrel{P\text{-a.e.}}{=} \{X_i^{(k)} < x < X_i^{(k+1)}\} \quad \text{for } k = 1, \dots, m-1, \\ \{X_i^*(x) = m\} &\stackrel{P\text{-a.e.}}{=} \{X_i^{(m)} < x\}. \end{aligned}$$

From (6) we can easily deduce

$$(7) \quad \{X_i^*(x) \geq k\} \stackrel{P\text{-a.e.}}{=} \{X_i^{(k)} < x\}$$

for $i = 1, \dots, n$ and $k = 0, 1, \dots, m$.

We now consider the Bernstein approximation to C . Let $x_1, \dots, x_n \in I$. Then

$$\begin{aligned} C_m(x_1, \dots, x_n) &= \sum_{i_1, \dots, i_n=1}^m C\left(\frac{i_1}{m}, \dots, \frac{i_n}{m}\right) b_{i_1,m}(x_1) \dots b_{i_n,m}(x_n) \\ &= \sum_{i_1, \dots, i_n=1}^m \left[\sum_{k_1=1}^{i_1} \dots \sum_{k_n=1}^{i_n} P\left(\frac{k_r-1}{m} < X_r < \frac{k_r}{m} : r = 1, \dots, n\right) \right. \\ &\quad \left. P(X_r^*(x_r) = i_r : r = 1, \dots, n) \right]. \end{aligned}$$

Since

$$\sum_{i_r=1}^m \sum_{k_r=1}^{i_r} = \sum_{k_r=1}^m \sum_{i_r=k_r}^m,$$

we have

$$\begin{aligned} C_m(x_1, \dots, x_n) &= \\ &= \sum_{k_1, \dots, k_n=1}^m \sum_{i_1=k_1}^m \dots \sum_{i_n=k_n}^m \left[P\left(\frac{k_r-1}{m} < X_r < \frac{k_r}{m} : r = 1, \dots, n\right) \right. \\ &\quad \left. P(X_r^*(x_r) = i_r : r = 1, \dots, n) \right] \\ &= \sum_{k_1, \dots, k_n=1}^m \left[P\left(\frac{k_r-1}{m} < X_r < \frac{k_r}{m} : r = 1, \dots, n\right) \right. \\ &\quad \left. P(X_r^*(x_r) \geq k_r : r = 1, \dots, n) \right] \\ &= \sum_{k_1, \dots, k_n=1}^m \left[P\left(\frac{k_r-1}{m} < X_r < \frac{k_r}{m} : r = 1, \dots, n\right) \right] \end{aligned}$$

$$P\left(X_r^{(k_r)} < x_r : r = 1, \dots, n\right) \Bigg]$$

where the last step follows from (7).

We now consider the cumulative distribution function of (Y_1, \dots, Y_n) .

$$\begin{aligned} &P(Y_r < x_r : r = 1, \dots, n) \\ &= \sum_{k_1, \dots, k_n=1}^m P(Y_r < x_r, (X_1, \dots, X_n) \in I_{m,k}^n : r = 1, \dots, n) \\ &= \sum_{k_1, \dots, k_n=1}^m \left[P\left(X_r^{(k_r)} < x_r : r = 1, \dots, n\right) \right. \\ &\quad \left. P\left(\frac{k_r - 1}{m} < X_r < \frac{k_r}{m} : r = 1, \dots, n\right) \right] \end{aligned}$$

by the definition of Y_r and the independence of the random variables. We see from this that

$$C_m(x_1, \dots, x_n) = P(Y_1 < x_1, \dots, Y_n < x_n). \quad \square$$

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Department of Mathematics, University of Central Florida, Orlando, FL 32816-1364
E-mail address: mtaylor@pegasus.cc.ucf.edu